RELATIONS FOR BERNOULLI–BARNES NUMBERS AND BARNES ZETA FUNCTIONS

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ABSTRACT. The *Barnes ζ-function* is

$$\zeta_n(z,x;\mathbf{a}) := \sum_{\mathbf{m}\in\mathbb{Z}^n_{\ge 0}} \frac{1}{(x+m_1a_1+\cdots+m_na_n)^z}$$

defined for $\mathrm{Re}(x)>0$ and $\mathrm{Re}(z)>n$ and continued meromorphically to $\mathbb{C}$. Specialized at negative integers $-k$, the Barnes ζ-function gives

$$\zeta_n(-k,x;\mathbf{a}) = \frac{(-1)^n k!}{(k+n)!} B_{k+n}(x;\mathbf{a})$$

where $B_k(x;\mathbf{a})$ is a *Bernoulli–Barnes polynomial*, which can be also defined through a generating function that has a slightly more general form than that for Bernoulli polynomials. Specializing $B_k(0;\mathbf{a})$ gives the *Bernoulli–Barnes numbers*. We exhibit relations among Barnes ζ-functions, Bernoulli–Barnes numbers and polynomials, which generalize various identities of Agoh, Apostol, Dilcher, and Euler.

1. INTRODUCTION

We define, as usual, the *Bernoulli numbers B_k* through the generating function

$$\frac{z}{e^z-1} = \sum_{k\ge 0} B_k \frac{z^k}{k!}\,. \tag{1}$$

A fundamental relation of Bernoulli numbers, known at least since Euler's time, is (for $n\ge 1$)

$$\sum_{j=0}^{n} \binom{n}{j} B_j B_{n-j} = -nB_{n-1} - (n-1)B_n\,. \tag{2}$$

Much more recently, multinomial generalizations of (2) were discovered by Agoh and Dilcher [1, 6]. They can be viewed as relations between Bernoulli numbers and *Bernoulli numbers $B_k^{(n)}$ of order n*, defined through

$$\left(\frac{z}{e^z-1}\right)^n = \sum_{k\ge 0} B_k^{(n)} \frac{z^k}{k!}\,. \tag{3}$$

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Dilcher and others proved generalized formulas relating $B_k^{(n)}$ with B_j. The first few are [6, p. 32]:

$$\begin{aligned} &B_k^{(2)} = -kB_{k-1} - (k-1)B_k \quad \text{(for any } k \geq 1), \\ (4)\quad &B_k^{(3)} = k(k-1)B_{k-2} + \tfrac{3}{2}k(k-2)B_{k-1} + \tfrac{1}{2}(k-1)(k-2)B_k \quad \text{(for any } k \geq 2), \\ &B_k^{(4)} = -6k(k-1)(k-2)B_{k-3} - 11k(k-1)(k-3)B_{k-2} - 6k(k-2)(k-3)B_{k-1} \quad \text{(for any } k \geq 3). \end{aligned}$$

Our first goal is to derive relations among *Bernoulli–Barnes numbers* $B_k(\mathbf{a})$, defined for a fixed vector $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n_{>0}$ through

$$\frac{z^n}{(e^{a_1 z} - 1)\cdots(e^{a_n z} - 1)} = \sum_{k \geq 0} B_k(\mathbf{a}) \frac{z^k}{k!} . \tag{5}$$

Note that, with (3) and (5), the Bernoulli–Barnes numbers and Bernoulli numbers are related as

$$B_k(\mathbf{a}) = \sum_{m_1 + \cdots + m_n = k} \binom{k}{m_1, \ldots, m_n} a_1^{m_1 - 1} \cdots a_n^{m_n - 1} B_{m_1} \cdots B_{m_n} .$$

Of course, one retrieves the Bernoulli numbers of order n with the special case $a_1 = a_2 = \cdots = a_n = 1$, and the Bernoulli numbers by further specializing $n = 1$. Our first main result is as follows.

Theorem 1. *For $n \geq 3$, $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n_{>0}$, and odd $k \geq n-2$,*

$$\sum_{j=2}^{n} \binom{n+j-4}{j-2} \frac{k!}{(k-n+j)!} \sum_{|I|=j} B_{k-n+j}(\mathbf{a}_I) = \begin{cases} -\frac{1}{4} & \text{if } n = k = 3, \\ 0 & \text{otherwise,} \end{cases}$$

where the inner sum is over all subsets $I \subseteq \{1, 2, \ldots, n\}$ of cardinality j, and $\mathbf{a}_I := (a_i : i \in I)$.

Corollary 2. *For $n \geq 3$ and odd $k \geq n-2$,*

$$\sum_{j=2}^{n} \binom{n+j-4}{j-2} \frac{k!}{(k-n+j)!} \binom{n}{j} B_{k-n+j}^{(j)} = \begin{cases} -\frac{1}{4} & \text{if } n = k = 3, \\ 0 & \text{otherwise,} \end{cases}$$

For example, for $n = 3, 4$ and odd $k \geq 1$ Theorem 1 gives the relations

$$\begin{aligned} B_k^{(3)} &= -\tfrac{3}{2}kB_{k-1}^{(2)} \qquad (k \geq 3), \\ B_k^{(4)} &= -2kB_{k-1}^{(3)} - k(k-1)B_{k-2}^{(2)} \qquad (k \geq 4). \end{aligned}$$

More generally, for any positive integer $n \geq 4$, Corollary 2 gives the following recurrence formula for the numbers $B_k^{(n)}$:

$$\binom{2n-4}{n-2} B_k^{(n)} = -\sum_{j=2}^{n-1} \binom{n+j-4}{j-2} \frac{k!}{(k-n+j)!} \binom{n}{j} B_{k-n+j}^{(j)} \qquad \text{for any odd } k \geq n-2. \tag{6}$$

The novelty of these relations as, e.g., compared with (4) is that they are between Bernoulli numbers of order higher than 1. We suspect that there are relations analogous to (6) for *even* k but leave the search for them as an open problem.

One of the significances of Bernoulli numbers lies in the fact that they are essentially evaluations of the *Riemann ζ-function* $\zeta(z) := \sum_{m \geq 1} m^{-z}$ (meromorphically continued to $\mathbb{C}$) at negative integers $-k$:

$$\zeta(-k) = -\frac{B_{k+1}}{k+1} .$$

Bernoulli–Barnes numbers appear in a similar fashion in relation with the *Barnes ζ-function*

$$\zeta_n(z,x;\mathbf{a}) := \sum_{\mathbf{m}\in\mathbb{Z}^n_{\geq 0}} \frac{1}{(x+m_1a_1+\cdots+m_na_n)^z}$$

defined for $\mathrm{Re}(x) > 0$ and $\mathrm{Re}(z) > n$ and continued meromorphically to $\mathbb{C}$ [7, 8, 9, 14, 17, 19]. Specialized at negative integers $-k$, the Barnes ζ-function gives

$$\zeta_n(-k,x;\mathbf{a}) = \frac{(-1)^n k!}{(k+n)!} B_{k+n}(x;\mathbf{a}) \tag{7}$$

where $B_k(x;\mathbf{a})$ is a *Bernoulli–Barnes polynomial*, defined through

$$\frac{z^n e^{xz}}{(e^{a_1z}-1)\cdots(e^{a_nz}-1)} = \sum_{k\geq 0} B_k(x;\mathbf{a}) \frac{z^k}{k!}\,. \tag{8}$$

Thus the Bernoulli–Barnes numbers are the special evaluations $B_k(\mathbf{a}) = B_k(0;\mathbf{a})$. It is clear that the Barnes zeta function is a multidimensional generalization of various Riemann–Hurwitz zetas functions; e.g., when $n = 1$ and $\mathbf{a} = (a)$, the function $\zeta(s;x,\mathbf{a})$ is the classical *Hurwitz zeta function* $a^{-s}\zeta(s;\frac{x}{a})$. Likewise, the Bernoulli–Barnes numbers and Bernoulli–Barnes polynomials extend the (generalized) Bernoulli numbers and polynomials to higher dimensions. Further generalizations of Bernoulli numbers and polynomials include [11, 18].

Our second main result expresses the Barnes zeta function in terms of Bernoulli–Barnes polynomials, Hurwitz zeta functions, and *Fourier–Dedekind sums* [3], defined as

$$\sigma_r(a_1,\ldots,\widehat{a}_j\ldots,a_d;a_j) := \frac{1}{a_j} \sum_{\lambda^{a_j}=1\neq\lambda} \frac{\lambda^r}{\prod_{1\leq k\neq j\leq n}(1-\lambda^{a_k})}\,.$$

Fourier–Dedekind sums generalize and unify many variants of (generalized) Dedekind sums; see, e.g., [16] or [4, Chapter 8].

Theorem 3. *Let $a_1,\ldots,a_n$ be pairwise coprime positive integers. Then*

$$\begin{aligned}\zeta(s;x,\mathbf{a}) \;&= \frac{(-1)^{n-1}}{(n-1)!}\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}B_{n-1-k}(x;\mathbf{a})\,\zeta(s-k;x)\\ &\quad+\sum_{j=1}^{n}a_j^{-s}\sum_{r=0}^{a_j-1}\sigma_{-r}(a_1,\ldots,\widehat{a}_j,\ldots,a_n;a_j)\,\zeta\left(s;\frac{x+r}{a_j}\right).\end{aligned}$$

Theorem 3 has many applications. Specializing s at negative integers gives, with the help of (7):

Corollary 4. *Let $a_1,\ldots,a_n$ be pairwise coprime positive integers. Then*

$$\begin{aligned}&\sum_{j=1}^{n}a_j^m\sum_{r=0}^{a_j-1}\sigma_{-r}(a_1,\ldots,\widehat{a}_j,\ldots,a_n;a_j)\,B_{m+1}\left(\frac{x+r}{a_j}\right)=\\ &\quad(-1)^{n-1}\frac{(m+1)!}{(m+n)!}B_{m+n}(x,\mathbf{a})+\frac{(-1)^{n-1}(m+1)}{(n-1)!}\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}B_{n-1-k}(x;\mathbf{a})\frac{B_{m+k+1}(x)}{m+k+1}.\end{aligned}$$

This is reminiscent of a *reciprocity law* for generalized Dedekind sums, due to Apostol [2]; this can be illustrated more easily in the case $n = 2$ and $\mathbf{a} = (a,b)$, for which Theorem 3 specializes to:

Corollary 5. *Let a,b be coprime positive integers. Then*

$$\zeta(s;x,(a,b)) = \frac{1}{ab}\zeta(s-1;x) + \left(1-\frac{x}{ab}\right)\zeta(s;x) - a^{-s}\sum_{r=0}^{a-1}\left\{\frac{b^{-1}r}{a}\right\}\zeta\left(s;\frac{x+r}{a}\right) - b^{-s}\sum_{r=0}^{a-1}\left\{\frac{a^{-1}r}{b}\right\}\zeta\left(s;\frac{x+r}{b}\right).$$

Again the specialization of s at negative integers gives, for $n=2$ and $\mathbf{a}=(a,b)$, using (7):

Corollary 6. *Let a,b be coprime positive integers. Then*

$$a^m\sum_{r=0}^{a-1}\left\{\frac{b^{-1}r}{a}\right\}B_{m+1}\left(\frac{x+r}{a}\right) + b^m\sum_{r=0}^{a-1}\left\{\frac{a^{-1}r}{b}\right\}B_{m+1}\left(\frac{x+r}{b}\right) = \frac{1}{m+2}B_{m+2}(x,(a,b)) + \frac{1}{ab}\frac{m+1}{m+2}B_{m+2}(x) + \left(\frac{x}{ab}-1\right)B_{m+1}(x).$$

This is a “polynomial generalization” of Apostol’s reciprocity law [2]

$$\frac{1}{m}\left(a^{m-1}s_m(a,b) + b^{m-1}s_m(b,a)\right) = \frac{B_{m+1}}{(m+1)ab} + \frac{1}{m(m+1)ab}(aB-bB)^{m+1}.$$

Here m is a positive integer, a and b are coprime, we use the umbral notation

$$(aB-bB)^{m+1} = \sum_{i=0}^{m+1}\binom{m+1}{i}(-1)^{m+1-i}a^i b^{m+1-i}B_i B_{m+1-i},$$

and

$$S_m(a,b) = \sum_{r=0}^{a-1}\left\{\frac{a^{-1}r}{b}\right\}B_m\left(\frac{r}{b}\right) \tag{9}$$

are the *Apostol–Dedekind sums*. The classical Dedekind sums [5, 16] are captured by the special case $m=1$. Thus in some sense, our study can be viewed as a bridge between Euler-type identities and Dedekind-type reciprocity laws.

Finally, we discuss the special case $\mathbf{a}=(1,\ldots,1)$ of Theorem 3. Denote

$$\zeta_n(s;x) = \zeta(s;x,(1,\ldots,1)),$$

the *Hurwitz zeta function of order n*. Since in this case the sums $\sigma_r(a_1,\ldots,\widehat{a}_j,\ldots,a_n;a_j)$ vanish, we obtain the following identity.

Corollary 7. *For any positive integer n,*

$$\zeta_n(s;x) = \frac{(-1)^{n-1}}{(n-1)!}\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}B_{n-1-k}^{(n)}(x)\,\zeta(s-k;x).$$

Specializing once more $s=-m$ at a negative integer gives:

Corollary 8. *For any positive integers n,m,*

$$B_{m+n}^{(n)}(x) = (m+n)\binom{m+n-1}{n-1}\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}B_{n-1-k}^{(n)}(x)\frac{B_{m+k+1}(x)}{m+k+1}.$$

Corollary 8 recovers once more Euler's identity (2) and Dilcher's results in [6]. We also note that in the above corollaries the coefficients of the polynomials $B^{(n)}_{n-1-k}(x)$ can be explicitly given by Stirling numbers of the first kind $s(n,k)$ as follows:

$$\binom{n-1}{n-1-k} B^{(n)}_{n-1-k}(x) = \sum_{m=0}^{n-1-k} \binom{m+k}{m} s(n,m+k+1)\, x^m \tag{10}$$

(see, e.g., [10, Equation (52.2.21)]).

2. Proof of Theorem 1

Our proof is based on identities of generating functions. Fix $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n_{>0}$ We define for a subset $I \subseteq \{1,2,\ldots,n\}$

$$f_I(z) := \frac{z^{|I|} e^{z \sum_{i\in I} a_i}}{\prod_{i\in I}(e^{a_i z}-1)} \qquad \text{and} \qquad F^{(j)}(z) := \sum_{|I|=j} f_I(z)\,.$$

Theorem 1 will follow from a relation of the even/odd parts of the functions $F^{(j)}(z)$, and to this extent we define

$$EO^{(j)}(z) := \begin{cases} \text{even part of } F^{(j)}(z) & \text{if } j \text{ is odd,} \\ \text{odd part of } F^{(j)}(z) & \text{if } j \text{ is even.} \end{cases}$$

Note that the even/odd part of $F^{(j)}(z)$ has the compact description

$$\sum_{|I|=j} \frac{z^{|I|}\left(e^{z\sum_{i\in I} a_i} \pm 1\right)}{2\prod_{i\in I}(e^{a_i z}-1)}\,.$$

Proposition 9. $\displaystyle\sum_{j=2}^{n} \binom{n+j-4}{j-2} (-z)^{n-j} EO^{(j)}(z) = \begin{cases} -\frac{1}{4}z^3 & \textit{if } n=3, \\ 0 & \textit{if } n \geq 4. \end{cases}$

Proof. The case $n=3$ is easily verified, so suppose $n \geq 4$. What the proposition is claiming in this case is that the function

$$F(z) := \sum_{j=2}^{n} \binom{n+j-4}{j-2} (-z)^{n-j} F^{(j)}(z) = \sum_{|I|\geq 2} \binom{n+|I|-4}{|I|-2} (-z)^{n-|I|} f_I(z)$$

is even, so that if suffices to prove that $F(z)$ equals

$$F(-z) = \sum_{|I|\geq 2} \binom{n+|I|-4}{|I|-2} z^{n-|I|} e^{-z\sum_{i\in I} a_i} f_I(z)\,,$$

i.e., that

$$F(z) - F(-z) = \sum_{|I|\geq 2} \binom{n+|I|-4}{|I|-2} z^{n-|I|} f_I(z) \left((-1)^{n-|I|} - e^{-z\sum_{i\in I} a_i}\right)$$

is zero. Written with the denominator $\prod_{i=1}^{n}(e^{a_i z}-1)$, the function $F(z)-F(-z)$ has the numerator

$$\sum_{|I|\geq 2} \binom{n+|I|-4}{|I|-2} z^n e^{z\sum_{i\in I} a_i} \left((-1)^{n-|I|} - e^{-z\sum_{i\in I} a_i}\right) \prod_{i\notin I}(e^{a_i z}-1)$$

and so we can rephrase our goal to proving that

$$\sum_{|I|\geq 2} \binom{n+|I|-4}{|I|-2} \left((-1)^{n-|I|} e^{z\sum_{i\in I} a_i} - 1\right) \prod_{i\notin I}(e^{a_i z}-1)$$

is zero. With $\prod_{i\notin I}(e^{a_i z}-1)=\sum_{J\subseteq \bar{I}}(-1)^{n-|I|-|J|}e^{z\sum_{i\in J}a_i}$, we can further rephrase our goal to proving that

$$\sum_{|I|\geq 2}\binom{n+|I|-4}{|I|-2}\sum_{J\subseteq\bar{I}}(-1)^{|J|}e^{z\sum_{i\in I\cup J}a_i}=\sum_{|I|\geq 2}\binom{n+|I|-4}{|I|-2}\sum_{J\subseteq\bar{I}}(-1)^{n-|I|-|J|}e^{z\sum_{i\in J}a_i}. \tag{11}$$

We will show that the coefficients of $e^{z\sum_{i\in K}a_i}$, for any $K\subseteq\{1,2,\dots,n\}$, on both sides of (11) are equal. This coefficient is on the left-hand side of (11) equal to

$$\sum_{j=0}^{|K|-2}(-1)^j\binom{|K|}{j}\binom{n+|K|-4-j}{|K|-2-j}.$$

The corresponding coefficient on the right-hand side of (11) is

$$\sum_{I\not\supseteq K}(-1)^{n-|I|-|K|}\binom{n+|I|-4}{|I|-2}=\sum_{j=2}^{n-|K|}(-1)^{n-j-|K|}\binom{n+j-4}{j-2}\binom{n-|K|}{j}.$$

Thus (11) is equivalent to

$$\sum_{j=0}^{k-2}(-1)^j\binom{k}{j}\binom{n+k-4-j}{k-2-j}=\sum_{j=2}^{n-k}(-1)^{n-j-k}\binom{n+j-4}{j-2}\binom{n-k}{j}.$$

But both sides equal $\binom{n-4}{k-2}$, as one can prove, e.g., by putting either side into a generating function for k. $\square$

Proof of Theorem 1. Recalling that $\mathbf{a}_I=(a_i : i\in I)$, we see that

$$f_I(z)=\frac{z^{|I|}e^{z\sum_{i\in I}a_i}}{\prod_{i\in I}(e^{a_i z}-1)}=\frac{(-z)^{|I|}}{\prod_{i\in I}(e^{-a_i z}-1)}$$

is the generating function for $(-1)^k B_k(\mathbf{a}_I)$, and we obtain for odd j

$$\begin{aligned}(-z)^{n-j}EO^{(j)}(z)&=(-1)^{n-1}\sum_{|I|=j}\sum_{\substack{k\geq 0\\ k+n-j \text{ odd}}}(-1)^k B_k(\mathbf{a}_I)\frac{z^{k+n-j}}{k!}\\ &=(-1)^{n-1}\sum_{|I|=j}\sum_{k\geq n-j \text{ odd}}\frac{k!}{(k-n+j)!}B_{k-n+j}(\mathbf{a}_I)\frac{z^k}{k!}\end{aligned}$$

and for even j

$$\begin{aligned}(-z)^{n-j}EO^{(j)}(z)&=(-1)^{n}\sum_{|I|=j}\sum_{\substack{k\geq 0\\ k+n-j \text{ odd}}}(-1)^k B_k(\mathbf{a}_I)\frac{z^{k+n-j}}{k!}\\ &=(-1)^{n-1}\sum_{|I|=j}\sum_{k\geq n-j \text{ odd}}\frac{k!}{(k-n+j)!}B_{k-n+j}(\mathbf{a}_I)\frac{z^k}{k!}.\end{aligned}$$

Theorem 1 follows by extracting the generating-function coefficients of the identity of Proposition 9. $\square$

We should remark that Theorem 1 was in part motivated by [13] in which Katayama proposed a three-term generalization of the reciprocity theorem for Dedekind–Apostol sums [2]; Apostol's theorem was a byproduct of another paper of Katayama [12]. Unfortunately, the main theorem of [13] is wrong; to make the central integral of the paper work, one has to use the integrand $\frac{z^{s-1}}{(e^{z/a}-1)(e^{z/b}-1)(e^{z/c}-1)}$ which, unfortunately, does not give rise to Dedekind–Apostol sums. However, using this integrand we discovered Theorem 1.

3. PROOF OF THEOREM 3

The function

$$p_A(t) := \#\left\{(k_1,\ldots,k_n) \in \mathbb{Z}_{\geq 0}^n : k_1 a_1 + \cdots + k_n a_n = t\right\},$$

which counts all partitions of t with parts in the finite set $A := \{a_1,\ldots,a_n\}$, is called a *restricted partition function*. For example, basic combinatorics gives

$$p_{\{1,\ldots,1\}}(t) = \binom{n-1+t}{n-1},$$

and a slightly less trivial example was proved by Popoviciu [15] (see also [4, Chapter 1]): for a and b coprime,

$$p_{\{a,b\}}(t) = \frac{t}{ab} + 1 - \left\{\frac{b^{-1}t}{a}\right\} - \left\{\frac{a^{-1}t}{b}\right\},$$

where $\{x\} = x - \lfloor x \rfloor$ denotes the fractional part of x, a^{-1} is computed mod b, and b^{-1} mod a.

The following theorem was proved in [3]; however, the authors of that paper did not realize the explicit role of Bernoulli–Barnes polynomials.

Theorem 10. *If $a_1,\ldots,a_n$ are pairwise coprime positive integers, then*

$$p_A(t) = \frac{(-1)^{n-1}}{(n-1)!} B_{n-1}(-t;(a_1,\ldots,a_n)) + \sum_{j=0}^{n} \sigma_{-t}\left(a_1,\ldots,\widehat{a_j},\ldots,a_n;a_j\right).$$

Proof. We give an outline of the proof. As in [3], we compute the residues of

$$F_t(z) = \frac{1}{z^{t+1}\prod_{i=1}^{n}(1-z^{a_i})}.$$

The residue at $z=0$ gives $p_A(t)$, whereas the residue at $z=1$ gives $\frac{(-1)^n}{(n-1)!} B_{n-1}(-t;(a_1,\ldots,a_n))$. Finally, if λ is a nontrivial a_ith root of unity,

$$\operatorname{Res}(F_t(z);z=\lambda) = -\frac{1}{a_i\lambda^t\prod_{j\neq i}(1-\lambda^{a_j})},$$

where the product runs over all $j=1,\ldots,n$ except $j=i$. Thus

$$\sum_{\lambda^{a_i}=1\neq\lambda} \operatorname{Res}(F_t(z);z=\lambda) = -\sigma_{-t}\left(a_1,\ldots,\widehat{a_i},\ldots,a_n;a_i\right).$$

The residue theorem completes the proof of Theorem 10. □

Proof of Theorem 3. Set $t = m_1a_1+\cdots+m_na_n$. We can rewrite the Barnes zeta function as follows:

$$\zeta(s;x,\mathbf{a}) = \sum_{m_1,\ldots,m_n\geq 0} \frac{1}{(x+m_1a_1+\cdots+m_na_n)^s} = \sum_{t\geq 0}\frac{p_A(t)}{(x+t)^s}.$$

By applying Taylor's Theorem to the function $t \mapsto B_{n-1}(-t;\mathbf{a})$ at $t=-x$,

$$B_{n-1}(-t;\mathbf{a}) = \sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}B_{n-1-k}(x;\mathbf{a})(x+t)^k,$$

and so with Theorem 10 we obtain

$$p_A(t) = \frac{(-1)^{n-1}}{(n-1)!}\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}B_{n-1-k}(x;\mathbf{a})(x+t)^k + \sum_{j=0}^{n}\sigma_{-t}\left(a_1,\ldots,\widehat{a_j},\ldots,a_n;a_j\right).$$

Hence

$$\zeta(s;x,\mathbf{a}) = \frac{(-1)^{n-1}}{(n-1)!}\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}B_{n-1-k}(x;\mathbf{a})\,\zeta(s-k;x) + \sum_{j=1}^{n}\sum_{t\geq 0}\frac{\sigma_{-t}(a_1,\dots,\widehat{a}_j,\dots,a_n;a_j)}{(x+t)^s}\,.$$

Note that $\sigma_{-t}\left(a_1,\dots,\widehat{a_j},\dots,a_n;a_j\right)$ depends only t mod a_j. Setting $t = ma_j + r$, $0 \le r \le a_j - 1$, we obtain

$$\zeta(s;x,\mathbf{a}) = \frac{(-1)^{n-1}}{(n-1)!}\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}B_{n-1-k}(x;\mathbf{a})\,\zeta(s-k;x)$$
$$+\sum_{j=1}^{n}\sum_{r=0}^{a_j-1}\sigma_{-r}(a_1,\dots,\widehat{a}_j,\dots,a_n;a_j)\sum_{m\geq 0}\frac{1}{(x+r+ma_j)^s}\,.$$

Writing $(x+r+ma_j)^{-s} = a_j^{-s}\left(\frac{x+r}{a_j}+m\right)^{-s}$ completes the proof of Theorem 3. □

4. The special case $\mathbf{a} = (a,1,1,\dots,1)$

In the special case $A = \{a,1,1,\dots,1\}$ (with n 1's), most of the terms in Theorem 10 disappear and we obtain

$$p_{\{a,1,1,\dots,1\}}(t) = \frac{(-1)^n}{n!}B_n(-t;(a,1,1,\dots,1)) + \sigma_{-t}\left(1,1,\dots,1;a\right). \tag{12}$$

On the other hand, we can apply [4, Theorem 8.8] to this special case; thus for $t = 1,2,\dots,a+n-1$,

$$\sigma_t\left(1,1,\dots,1;a\right) = \frac{(-1)^{n-1}}{n!}B_n(t;(a,1,1,\dots,1))\,.$$

Since $\sigma_t\left(1,1,\dots,1;a\right)$ only depends on t mod a, this range for t is enough to determine $\sigma_t\left(1,1,\dots,1;a\right)$:

$$\sigma_t\left(1,1,\dots,1;a\right) = \begin{cases} \frac{(-1)^{n-1}}{n!}B_n(t \bmod a;\,(a,1,1,\dots,1)) & \text{if } t \not\equiv 0 \bmod a,\\ \frac{(-1)^{n-1}}{n!}B_n(a;\,(a,1,1,\dots,1)) & \text{if } t \equiv 0 \bmod a.\end{cases} \tag{13}$$

For the case $t \equiv 0$ mod a we can also use [4, Theorem 8.4] which gives

$$\sigma_0\left(1,1,\dots,1;a\right) = 1 - \frac{(-1)^n}{n!}B_n(0;(a,1,1,\dots,1))\,.$$

(An easy way to see that our two formulations of $\sigma_0\left(1,1,\dots,1;a\right)$ are equivalent is through the difference formula

$$B_m\left(x+a_0;(a_0,a_1,\dots a_n)\right) - B_m\left(x;(a_0,a_1,\dots a_n)\right) = m\,B_{m-1}\left(x;(a_1,a_2,\dots a_n)\right)$$

and then specializing this to $m = n$, $x = 0$, $a_0 = a$, and $a_1 = a_2 = \cdots = a_n = 1$.) Substituting this back into (12) gives, with $\chi_a(t) := 1$ if $a|t$ and $\chi_a(t) := 0$ otherwise:

Proposition 11. $p_{\{a,1,1,\dots,1\}}(t) = \frac{(-1)^n}{n!}\left(B_n(-t;(a,1,1,\dots,1)) - B_n(t \bmod a;\,(a,1,1,\dots,1))\right) + \chi_a(t)\,.$

Using (13) and Theorem 3 we obtain:

Proposition 12. *Let* $\mathbf{a} = (a,1,1,\dots,1)$, *where a is a positive integer. Then*

$$\zeta(s;x,\mathbf{a}) = \frac{(-1)^{n-1}}{(n-1)!}\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}B_{n-1-k}(x;\mathbf{a})\,\zeta(s-k;x) + \frac{(-1)^{n-1}}{n!}a^{-s}\sum_{r=1}^{a}B_n(r;\mathbf{a})\,\zeta\left(s;1+\frac{x-r}{a}\right).$$

Specializing $s = -m$ at negative integers gives, by Proposition 12 with the help of (7), the following formula.

Corollary 13. *Let* $\mathbf{a} = (a, 1, 1, \ldots, 1)$*, where a is a positive integer. Then*

$$\frac{m!n!}{(m+n)!} B_{m+n}(x,\mathbf{a}) = \sum_{k=0}^{n-1} (-1)^k (k+1) \binom{n}{k+1} B_{n-1-k}(x;\mathbf{a}) \frac{B_{m+k+1}(x)}{m+k+1} + a^m \sum_{r=1}^{a} B_n(r;\mathbf{a}) \frac{B_{m+1}\left(1+\frac{x-r}{a}\right)}{m+1}.$$

5. Difference, symmetry and recurrence formulas for $B_m(x;\mathbf{a})$

We conclude by giving various formulas for $B_m(x;\mathbf{a})$, starting with the following difference formula.

Theorem 14. *For* $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n_{\geq 0}$*, we have the difference formula*

$$(-1)^m B_m(-x;\mathbf{a}) - B_m(x;\mathbf{a}) = m! \sum_{k=0}^{n-1} \sum_{|I|=k} \frac{B_{m-n+k}(x;\mathbf{a}_I)}{(m-n+k)!},$$

with $B_m(x,\mathbf{a}_I) = x^m$ *if* $I = \emptyset$*. Furthermore,*

$$B_m(x + \sum_{i=1}^{n} a_i;\mathbf{a}) = (-1)^m B_m(-x;\mathbf{a}).$$

Proof. By use of the identity

$$\sum_{I\subset\{1,\ldots,n\}} \prod_{i\in\bar{I}} \left(e^{a_i t} - 1\right) = e^{(a_1+\cdots+a_n)t},$$

we obtain the formula

$$\frac{t^n e^{(x+\sum_{i=1}^{n} a_i)t}}{\prod_{i=1}^{n} \left(e^{a_i t} - 1\right)} = \sum_{I\subset\{1,\ldots,n\}} \frac{t^n e^{xt}}{\prod_{i\in I} \left(e^{a_i t} - 1\right)}, \tag{14}$$

where $\bar{I} = \{1,\ldots,n\} \backslash I$. On the other hand, we have the equality

$$\frac{t^n e^{(x+\sum_{i=1}^{n} a_i)t}}{\prod_{i=1}^{n} \left(e^{a_i t} - 1\right)} = \frac{(-t)^n e^{xt}}{\prod_{i=1}^{n} \left(e^{-a_i t} - 1\right)}. \tag{15}$$

Therefore, by (14) and (15),

$$\sum_{I\subset\{1,\ldots,n\}} \frac{t^n e^{xt}}{\prod_{i\in I} \left(e^{a_i t} - 1\right)} = \frac{(-t)^n e^{xt}}{\prod_{i=1}^{n} \left(e^{-a_i t} - 1\right)}.$$

Together with (8) this completes the proof. □

Our next result is a symmetry formula.

Theorem 15. *Let* $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n_{\geq 0}$ *with* $A := a_1 + \cdots + a_n > 0$*. Then for any positive integers* $l, m \geq 1$*, we have*

$$(-1)^m \sum_{k=0}^{m} \binom{m}{k} A^{m-k} B_{l+k}(x;\mathbf{a}) = (-1)^l \sum_{k=0}^{l} \binom{l}{k} A^{l-k} B_{m+k}(-x;\mathbf{a}), \tag{16}$$

and

$$\begin{aligned}
&\frac{(-1)^m}{m+l+2}\sum_{k=0}^{m}\binom{m+1}{k}(l+k+1)A^{m+1-k}B_{l+k}(x;\mathbf{a}) \\
&\quad+\frac{(-1)^l}{m+l+2}\sum_{k=0}^{n}\binom{l+1}{k}(l+k+1)A^{l+1-k}B_{m+k}(-x;\mathbf{a}) \\
&=(-1)^{m+1}B_{l+m+1}(x;\mathbf{a})+(-1)^{l+1}B_{n+m+1}(-x;\mathbf{a}).
\end{aligned} \tag{17}$$

Proof. Observe that from (8) we obtain

$$\frac{d}{dx}B_{l+1}(x;\mathbf{a})=(l+1)B_l(x;\mathbf{a})$$

and by applying the operator $\frac{d}{dx}$ to (16), this implies (17).

Now we prove (16). Consider the generating function

$$\begin{aligned}
&\sum_{m\geq 0}\sum_{l\geq 0}\left((-1)^m\sum_{k=0}^{m}\binom{m}{k}A^{m-k}B_{l+k}(x;\mathbf{a})\right)\frac{y^m}{m!}\frac{z^l}{l!} \\
&=\sum_{l\geq 0}\sum_{k\geq 0}A^{-k}B_{l+k}(x;\mathbf{a})\frac{z^l}{l!}\sum_{m=k}^{\infty}(-1)^m\binom{m}{k}\frac{(Ay)^m}{m!} \\
&=\sum_{l\geq 0}\sum_{k\geq 0}B_{l+k}(x;\mathbf{a})\frac{z^l}{l!}\frac{(-y)^k}{k!}e^{-Ay}=e^{-Ay}\sum_{i\geq 0}\sum_{k=0}^{i}B_i(x;\mathbf{a})\frac{z^{i-k}}{(i-k)!}\frac{(-y)^k}{k!} \\
&=\sum_{l\geq 0}\sum_{k\geq 0}B_{l+k}(x;\mathbf{a})\frac{z^l}{l!}\frac{(-y)^k}{k!}e^{-Ay}=e^{-Ay}\sum_{i\geq 0}\sum_{k=0}^{i}\frac{B_i(x;\mathbf{a})}{i!}\binom{i}{k}z^{i-k}(-y)^k \\
&=e^{-Ay}\sum_{i\geq 0}B_i(x;\mathbf{a})\frac{(z-y)^i}{i!}=e^{-Ay}\frac{(z-y)^n e^{x(z-y)}}{\prod_{i=1}^{n}\left(e^{a_i(z-y)}-1\right)}=\frac{(z-y)^n e^{x(z-y)}}{\prod_{i=1}^{n}\left(e^{a_iz}-e^{a_iy}\right)}.
\end{aligned}$$

Similarly the generating function is also equal to

$$\sum_{m\geq 0}\sum_{l\geq 0}\left((-1)^l\sum_{k=0}^{l}\binom{l}{k}A^{l-k}B_{m+k}(-x;\mathbf{a})\right)\frac{y^m}{m!}\frac{z^l}{l!}=\frac{(z-y)^n e^{x(z-y)}}{\prod_{i=1}^{n}(e^{a_iz}-e^{a_iy})}.$$

□

Specializing Theorem 15 at $l=m$ we obtain a recurrence formula for the polynomials

$$P_m(x):=(m+1)A^{-m}\Big(B_m(-x;\mathbf{a})+B_m(x;\mathbf{a})\Big)$$

as follows.

Corollary 16. *For any positive integer $m\geq 1$,*

$$P_{2m+1}(x)=-\frac{2m+1}{2(m+1)}\sum_{k=0}^{m}\binom{m+1}{k}P_{m+k}(x).$$

In the case $n=1$, $a_1=1$, the polynomials $P_m(x)$ are reduced to $(m+1)\Big(B_m(-x)+B_m(x)\Big)$.

Moreover, for the Bernoulli–Barnes numbers we obtain the following recurrence formula.

Theorem 17. *For any positive integer $m \geq 1$,*

$$B_{2m+1}(\mathbf{a}) = -\frac{1}{2(m+1)} \sum_{k=0}^{m} \binom{m+1}{k} (m+k+1) A^{m+1-k} B_{m+k}(\mathbf{a})$$

and

$$B_{2m}(\mathbf{a}) = -\frac{1}{(m+1)(2m+1)} \sum_{k=0}^{m} \binom{m+1}{k} (m+k+1) A^{m-k} B_{m+k}(\mathbf{a}) - (2m)!\, A^{-1} \sum_{k=0}^{n-1} \sum_{|I|=k} \frac{B_{2m+1-n+k}(\mathbf{a}_I)}{(2m+1-n+k)!}.$$

Note that for $n = 1$, the above results specialize to the well-known difference, symmetry and recurrence concerning the ordinary Bernoulli numbers and polynomials.

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